Indian Statistical Institute
M. Math 2nd year
Academic year 2022-2023
Midterm Examination
Course: Special Topics in Geometry: Harmonic maps
22 - 02 - 23
2 hours

- *Answer as many questions as you can.*
- *You may use results proved in class, but make sure to state them clearly.*
- *Maximum marks is 40.*

1. Let $M_n(\mathbb{R})$ denote the space of $n \times n$ real matrices equipped with the operator norm, and $GL_n(\mathbb{R})$ the subset of invertible $n \times n$ matrices.

   (a) Show that $GL_n(\mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.

   (b) Show that the map $f : GL_n(\mathbb{R}) \to M_n(\mathbb{R}), A \mapsto A^{-1}$, is differentiable and compute its derivative $Df_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ at every $A \in GL_n(\mathbb{R})$. (2 + 8 = 10 marks)

2. Let M be a smooth manifold and let $\gamma_0, \gamma : [0, 1] \to M$ be closed curves in M. We say that γ_0, γ_1 are homotopic in M if there is a continuous map $\sigma : [0, 1] \times [0, 1] \to M$ such that $\sigma(s, 0) = \gamma_0(s), \sigma(s, 1) = \gamma_1(s)$ for all $s \in [0, 1]$, and $\sigma(0, t) = \sigma(1, t)$ for all $t \in [0, 1]$. Suppose there is a smooth homotopy of closed curves $\sigma : [0, 1]^2 \to M$ between γ_0 and γ_1.

   (a) Show that the boundary of the singular 2-cube σ is given by $\partial\sigma = \gamma_0 - \gamma_1$.

   (b) Hence show that if ω is a closed 1-form on M then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$. (4 + 4 = 8 marks)

3. Let M, N be a smooth manifolds and let $H : M \times [0,1] \to N$ be a smooth homotopy between smooth maps $f, g : M \to N$, so that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in M$. Let $P : C_k(M) \to C_{k+1}(N)$ be the associated prism operator, such that for any singular k-cube $c : [0,1]^k \to M$ in M, the singular $(k+1)$-cube $Pc : [0,1]^{k+1} \to N$ in N is defined by

$$(Pc)(t, x_1, \ldots, x_k) = H(c(x_1, \ldots, x_k), t) \ , \ (t, x_1, \ldots, x_k) \in [0,1]^{k+1}.$$

(a) Show that

$$\partial Pc = (g \circ c - f \circ c) - P(\partial c)$$

for all singular k-cubes c in M.

(b) Show that there is a linear map $P^* : \Omega^{k+1}(N) \to \Omega^k(M)$ from $(k+1)$ forms on N to k forms on M such that

$$\int_{Pc} \omega = \int_c P^* \omega$$

for all singular k-cubes c in M and for all $(k+1)$ forms ω on N.

(c) Suppose that $M = N$, $g = \mathrm{id}_M$, and f is a constant map. Show that any closed form on M is exact. (6+6+6 = 18 marks)

4. Let E be a smooth vector bundle of rank k over a smooth n-manifold M. Let ∇ be a connection on E and let $\tilde{\nabla}$ denote the induced connection on the dual bundle E^*. Let $\gamma : [0,1] \to M$ be a smooth curve, and let $\frac{D}{dt}$ and $\frac{\tilde{D}}{dt}$ denote covariant differentiation along the curve γ for the bundles E and E^* respectively.

(a) Let s be a section of E along γ and let α be a section of E^* along γ. Show that

$$\frac{d}{dt}\alpha(t)(s(t)) = \frac{\tilde{D}\alpha}{dt}(s(t)) + \alpha(t)\left(\frac{Ds}{dt}\right).$$

(b) Let $P : E_{\gamma(0)} \to E_{\gamma(1)}$ and $\tilde{P} : E^*_{\gamma(0)} \to E^*_{\gamma(1)}$ be the parallel transport maps along γ for the bundles E and E^* respectively. Show that $\tilde{P} = (P^*)^{-1}$. (6+6 = 12 marks)